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Parameter-Plane Analysis for Large-Scale Systems

Sherman M. Seltzer*
Control Dynamics Company, Huntsville, Ala.
Bernard A. Asner Jr.†
The University of Dallas, Irving, Texas
and
Robert L. Jackson‡
TRW Defense and Space Systems Group, Redondo Beach, Calif.

The parameter-plane analysis technique recently has been extended to enable the analyst to portray stability regions for large-scale systems. In this paper the previous restriction on order of the system to be analyzed has been relaxed by implicitly assuming a model of the system. The technique is used to show that a design point established using the positivity concept lies well within the desired stable region of the selected parameter plane. This methodology is applied to the control of a model of a large space structure, assuring global stability despite modal truncation and poor knowledge of the plant.

I. Introduction

THE history of the continuous time-domain revision of the parameter-plane technique is well documented in Ref. 1 and updated in Ref. 2. Briefly, the technique had its origin with Vishnegradsky in 1876. In 1949, Neimark generalized Vishnegradsky's approach to permit the decomposition of a two-parameter domain (D) describing an nth order system into stable and unstable regions. The interested reader is directed to Refs. 1 and 2.

In the past, the authors have applied the parameter-plane technique to the design and analysis of control systems for large satellites. For example, it was successfully employed in the design of Skylab, the world's first orbiting, manned space station.[3] It has subsequently been used to prescribe structural stability constraints for large, flexible spacecraft.[4] In this paper a stability analysis is performed on a large-scale dynamic system (using the parameter-plane technique). Instead of deriving the characteristic equation, the original equations of motion are manipulated, leaving the tedious tasks of determinant evaluation to the computer. In a previous paper by the authors, the general technique was exploited for an nth order system, where n was restricted to be less than 100 (or so) for medium-size computers.[2] In the present case this condition on the order is relaxed by implicitly assuming a model of the system. The assumed structural model is given as a linear system of second-order differential equations which may be generated using a finite-element method (such as NASTRAN). Applying "modern" control theory, it is further assumed that a controller has been designed to estimate the states of the system and that the designers have at their disposal the optimal control law for the closed-loop system.

The parameter-plane technique focuses on two parameters at a time. For example, application of the method generates the stability boundaries of the closed-loop system as a function of these two selected parameters. Thus one can see the effect of variations of these parameters and determine how much they may vary before the system becomes unstable. If the parameters are elements of the feedback matrix, then it is shown how to reduce the computational task to one of evaluating matrices whose order is the same as the size of the estimator plant. Thus the size of the dynamics of the plant does not introduce any theoretical difficulties.

The results show that it is computationally feasible to generate a parameter-plane portrayal of stability conditions for structural modes whose control laws are derived using "modern" control theory. To substantiate this conclusion, a model of a large space system is presented. The system portrayed includes highly flexible solar arrays. The model is derived in the paper. Its dynamics are presented with 3 degrees of freedom, and the states are estimated with an eight-dimensional estimator. A design point is established using a positivity concept.[5] Finally, the parameter-plane technique is applied to portray the stability region. It is shown that the positivity concept produces a candidate design point located well within the desirable region of the parameter plane.

II. Description of Mathematical Model

In this paper the previous restriction of system order n is relaxed by implicitly assuming a mathematical model. The assumed structural model is given as a linear system of second-order differential equations which, for example, is generated using a finite-element method such as NASTRAN. Using modern control theory, one may further assume a controller has been designed to estimate the states of the system and that the optimal control law for the closed-loop system is available. In Sec. IV the description and derivation of a specific model is developed. For convenience, an alternate generic form of the derived system equations is given below:

$$\ddot{\eta}+\Omega^2\eta+\Phi^T e(\Phi^T e)^T(K_r\dot{\eta}+K_p\eta)+\Phi^T EK_c\hat{z}=0 \quad (1)$$

$$\dot{\hat{z}}-(A-G_cC_c-B_cK_c)\hat{z}-G_cE^T\Phi\dot{\eta}=0 \quad (2)$$

With the exception of the scalars K_r and K_p in the rigid body feedback loop, all symbols in the equations are understood to represent vectors and matrices. Most of these symbols are precisely defined in Sec. IV and in the block diagram of Fig. 4. Exceptions are the base vector e and the permutation matrix E. In the specific model $e=e_{15}$, where state 15 is the rigid body state, and $E=(e_2e_5e_{10}e_{13})$, where states 2, 5, 10, and 13 comprise the "feedback" states of the system.

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*Consulting Engineer. Associate Fellow AIAA.
†Chairman, Department of Mathematics.
‡Member, Technical Staff, Control and Sensor Systems Laboratory.

Equation (1) describes the dynamics of the plant in normal modal coordinates (η); and Eq. (2) is the model of the estimator dynamics (z). It is assumed that the dimension of the plant is relatively large and the dimension of the estimator is relatively small. Under these assumptions, and by centering interest on the matrix K_c, one can manipulate the characteristic equation of the closed-loop system to a form where it is only necessary to evaluate determinants whose orders are the size of the estimator plant.

Introducing the following notation,

$$u_R(s) = -(K_r s + K_p)$$

$$b = \Phi^T e$$

$$A = A_c - G_c C_c$$

$$X = s^2 I + \Omega^2 - u_R(s) bb^T$$

$$Y = \Phi^T E K_c$$

$$Z = -sG_c E^T \Phi$$

$$W = sI - A + B_c K_c$$

$$U(s) = sI - A$$

$$V(s) = B_c - ZX^{-1}\Phi^T E$$

allows one to write and expand the characteristic equation as

$$\det\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \det(X)\det(W - ZX^{-1}Y) = 0 \tag{3}$$

Since the parameters of interest for the example at hand are not included in the X matrix, it suffices (for a parameter-plane analysis) to consider the reduced determinant

$$\det(W - ZX^{-1}Y) = 0 \tag{4}$$

But

$$\begin{aligned} W - ZX^{-1}Y &= (sI - A) + B_c K_c + sG_c E^T \Phi X^{-1} \Phi^T E K_c \\ &= (sI - A) + (B_c + sG_c E^T \Phi X^{-1} \Phi^T E) K_c \\ &= U(s) + V(s) K_c \end{aligned} \tag{5}$$

so

$$\det(W - ZX^{-1}Y) = \det(U + VK_c) \tag{6}$$

An expression for the inverse of X is given by

$$X^{-1} = \left[I + \frac{u_R(s)(s^2 I + \Omega^2)^{-1} bb^T}{1 - u_R(s) b^T (s^2 I + \Omega^2)^{-1} b}\right](s^2 I + \Omega^2)^{-1} \tag{7}$$

Since $(s^2 I + \Omega^2)$ is a diagonal matrix, there is no need for a general purpose matrix inversion routine to find the inverse of X.

If the designer is interested in any of the parameters in the X matrix, then the equations should be reformulated to isolate these effects. This may be done as shown in Ref. 2.

III. The Parameter-Plane and Positivity Techniques

Brief descriptions of the parameter-plane technique and the positivity technique are included in this section to provide continuity to the paper. The references provide detailed background for the interested reader.

Parameter-Plane Technique

The Nyquist technique assumes that all the variables in the system, with the exception of one, are known. By varying the frequency and examining the resulting plot, one obtains stability criteria in terms of the unknown parameter. In contrast, the parameter-plane technique focuses on two parameters in the system equations. For a given $s = \sigma + j\omega$, one determines values of the two unknown parameters which satisfy the characteristic equation. A typical analysis fixes σ and allows ω to vary over specified ranges. The results are then plotted with one parameter as the abscissa and the other as the ordinate. If, for example, $\sigma = 0$, then the Euclidean plane is decomposed into disjoint sets, such that a point in the interior of each set implies the number of roots of the characteristic equation with positive real parts will remain unchanged. With a particular design point in mind, one can then induce robustness and parameter sensitivity by "measuring the size" of the stability region.

Positivity Technique

The design for the nominal controller is based on the positivity of operators techniques detailed in Refs. 5 and 6. This method can be applied to the control of large space structures to assure global stability despite modal truncation and poor knowledge of the plant. Figure 1 shows a block diagram of a closed-loop system where $G(s)$ is the plant and $H(s)$ represents the controller. The theorem developed in Ref. 5 states that if $G(s)$ and $H(s)$ are square transfer matrices, then the system S is asymptotically stable if at least one of the transfer matrices is strictly positive real and the other is positive real. [A square transfer matrix $Z(s)$ is called positive real if $Z(s)$ has real elements for real s; $Z(s)$ has elements which are analytical for $\mathrm{Re}(s) > 0$; $Z^*(s) + Z(s)$ is non-negative definite for $\mathrm{Re}(s) > 0$ (* denotes the complex conjugate transpose). A square transfer matrix $Z(s)$ is strictly positive real if $Z(s)$ has real elements for real s; $Z(s)$ has elements which are typical for $\mathrm{Re}(s) \geq 0$; $Z^*(j\omega)$ is positive definite for all real ω]. If the plant is energy dissipative (as many LSS are) and ideal collocated actuators and rate sensors are used, then the plant is positive real. If the plant is not positive real, then a technique called embedding may be used. This is a block diagram transformation used to transform both G and H to positive real transfer matrices by putting a further constraint on the controller.[6]

IV. Design Application

In this section a model of an example of a large space structure is derived. A control system is then designed for that model, using the positivity technique. This is followed by an application of the parameter-plane technique to show that the control system design point lies well within the stable region for the system.

Model Derivation

A large flexible communication satellite was used as an example of the application of the techniques of this paper. Figure 2 shows the spacecraft, with two large flexible solar arrays on the sides of the rigid main body. Structural vibration control of the out-of-plane modes of the array is

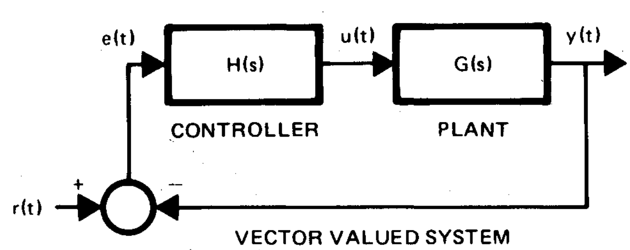

Fig. 1 System S (vector valued).

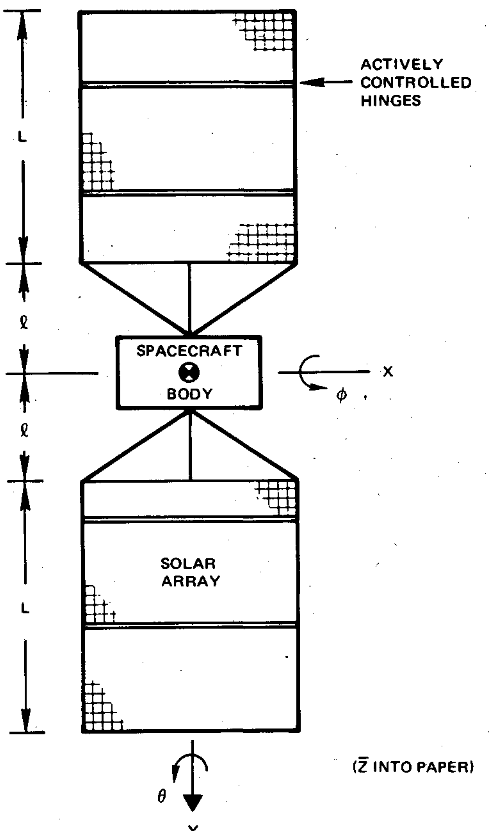

Fig. 2 Single-axis spacecraft model.

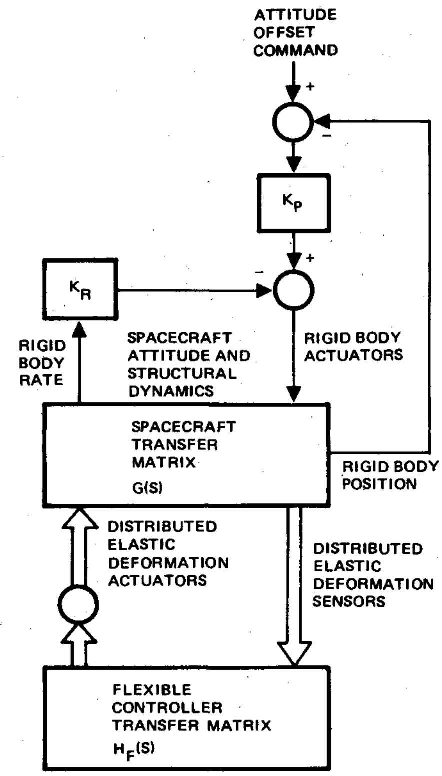

Fig. 3 Block diagram of control system.

accomplished using four hinge motors, applying relative torques between the panels. Relative rotations and rates are sensed by collocated angle encoders or tachometers. The rigid body attitude is controlled using reaction wheels located on the main body in conjunction with gyros to sense attitude.

Finite-element methods were used to generate a mathematical model for describing the dynamics of the spacecraft. For simplicity, the model is limited to two degrees of freedom: rotation about the roll axis x and translation along the yaw axis z. Each element of the model has a finite dimension and associated mass and inertia. Thus a dynamic equation describing the motion of each element can be derived as

$$M\ddot{q}+Kq=F \tag{8}$$

where q is the vector composed of relative rotations between elements and the rigid body translation and roll attitude, M is the mass matrix, K is the stiffness matrix, and F is the force/torque matrix.

Equation (8) is converted to modal form using the transformation

$$q=\Phi\eta \tag{9}$$

where Φ is the matrix of normalized eigenvectors (modal matrix). Substituting Eq. (9) into Eq. (8) and simplifying, one obtains

$$\ddot{\eta}+\Omega^2\eta=\Phi^t F \tag{10}$$

where Ω is the diagonal eigenvalue matrix (in this case the modal frequencies).

The outputs for this example will correspond to the displacements/rates (relative rotations and attitude) at the selected sensor locations:

$$y=C_1q+C_2\dot{q} \tag{11}$$

or in modal coordinates

$$y=C_1\Phi\eta+C_2\Phi\dot{\eta} \tag{12}$$

Similarly the F (force) matrix corresponds to actuator forces (moments) at selected stations. By transforming Eq. (10) one can arrive at the familiar linear state representation,

$$\dot{x}=Ax+Bu \qquad y=Cx \tag{13}$$

where the state vector x is defined as

$$x=(\eta_1,\dot{\eta}_1,\ldots,\eta_n,\dot{\eta}_n\ldots)^T$$

Control System Design

The nominal control system design approach is based on the positivity of operators concept as presented in Ref. 5. The methodology had to be altered slightly to accommodate the rigid body attitude control required for this example. Thus the control system design was performed in two parts as illustrated in Fig. 3. The inner loop uses the positivity design methodology to attenuate structural vibration. The outer loop uses a conventional rate plus position (PD) feedback control system, but it is, in fact, based on the positivity design approach.

Rigid Body Control Design

A position plus rate feedback law can be mathematically represented to satisfy the positivity condition, assuring stability of the system. To assure that the plant (spacecraft) transfer matrix remains positive real, only attitude rate is used as a plant output. Since rigid body attitude control demands position feedback the rigid body control law $H_R(s)$ contains an integrator, feeding back rate plus integrated rate (which is position). $H_R(s)$ is then given by

$$H_R(s)=K_R+K_p/s \qquad (K_R,K_p>0) \tag{14}$$

and, clearly, $H_R(s)$ is strictly positive real. Thus both the plant and the controller are positive, and stability is assured.

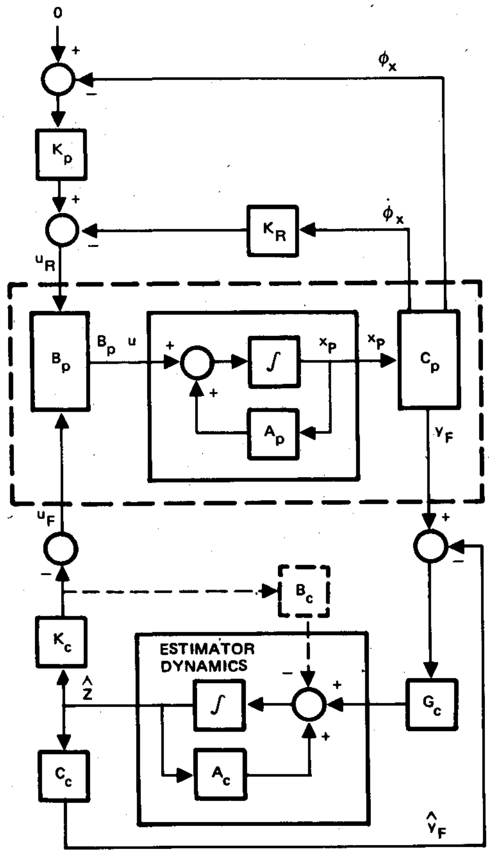

Fig. 4 Detailed control system diagram.

The gains K_p and K_R are chosen so that the equivalent rigid body second-order system

$$\ddot{\phi}+2\zeta\omega\dot{\phi}+\omega_n^2 c=m_n^2 c_c \tag{15}$$

has the desired damping and natural frequency characteristics. Thus

$$K_p=\omega_n^2 Ix \qquad K_R=2\zeta\omega_n I_x$$

Structural Controller

The structural vibration controller is now designed as outlined in the example in Ref. 6. Figure 4 shows the vibration controller along with the rigid body portion. The estimator gain matrix G_c is first chosen using LQG techniques with emphasis on reducing observation spillover. The feedback gain matrix K_c is determined such that the controller, $H(s)$, is strictly positive real. This is accomplished by solving the Lyapunov equation,

$$P\Gamma+\Gamma^T P=Q \tag{16}$$

for P, where $\Gamma=A_c-G_cC_c$ and Q is a positive definite matrix chosen to yield good performance.[5,7] The feedback gain matrix is then computed as

$$K_c^T=PG_c \tag{17}$$

completing the controller design. A summary of the controller system matrices is shown in Table 1.

Application of Parameter-Plane Technique

Referring to Sec. II, one finds the characteristic equation to be equivalent to [see Eq. (6)]

$$\det(U(s)+V(s)K_c)=0 \tag{18}$$

Table 1 Nominal controller system matrices

AC

0.00	1.00E+00	0.00	0.00	0.00	0.00	0.00	0.00
-5.92E-01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	1.00E+01	0.00	0.00	0.00	0.00
0.00	0.00	-1.34E+00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	1.00E+00	0.00	0.00
0.00	0.00	0.00	0.00	-2.37E+01	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00E+00
0.00	0.00	0.00	0.00	0.00	0.00	-2.55E+01	0.00

CC

0.00	-8.60E-04	0.00	-1.50E-03	0.00	2.28E-02	0.00	-2.38E-02
0.00	-4.28E-03	0.00	-6.72E-03	0.00	2.05E-02	0.00	-1.94E-02
0.00	1.80E-03	0.00	-3.04E-03	0.00	-3.42E-02	0.00	-3.32E-02
0.00	5.68E-03	0.00	-8.34E-03	0.00	4.36E-03	0.00	6.70E-03

BC

-2.13E+02	2.44E+02	-1.54E+01	-1.38E+02
2.43E+02	-2.78E+02	2.00E+01	1.73E+02
-4.55E+01	5.63E+01	9.29E+00	4.87E+01
1.47E+02	-1.00E+02	-2.25E+01	-1.11E+02
-1.98E-01	-8.54E-01	6.13E-01	1.11E-01
4.43E+01	-5.40E-01	-2.05E+01	6.26E+00
2.02E-01	7.16E-01	5.18E-01	-3.79E-01
-4.36E+01	2.22E+00	-2.64E+01	1.13E+01

KC

1.23E+01	1.63E+02	1.82E+00	6.49E+01	-1.74E+01	1.21E+01	1.65E+01	-1.18E+01
-1.42E+01	-1.86E+02	-2.66E+00	-7.93E+01	1.48E+01	-7.57E-01	-1.40E+01	1.12E+00
1.55E+00	1.37E+01	-1.46E+00	-9.78E+00	2.32E+00	-7.40E+00	3.15E+00	-6.00E+00
1.20E+01	1.17E+02	-8.72E+00	-4.79E+01	-6.12E+00	1.87E+00	3.27E+00	2.75E+00

K_p = 16630.5 $\qquad$ K_R = 13234

For the example at hand this constitutes evaluating an 8×8 determinant.

In the gain matrix K_c appearing in Eq. (18), one may denote the two parameters of interest by (α,β) and assume they are in (row, column) (I_α,J_α), and (I_β,J_β) respectively. The first task is to isolate these parameters by using elementary operations for expanding a determinant. Without changing the value of the determinant and without loss of generality, the rows and columns of $U+VK_c$ may be permuted to move the parameters α and β to the lower-right 2×2 submatrix. If the parameters are on the same row or column, they will be moved to locations (7,8) and (8,8). Otherwise they will be moved to locations (7,7) and (8,8). In the discussion that follows, this permutation is assumed to have been accomplished, and for clarity the notation is not altered.

Denote by small Latin letters the column vectors of the matrices so that

$$U=(u_1u_2\dots u_8) \qquad \in C^{8\times 8}$$

$$V=(v_1v_2v_3v_4) \qquad \in C^{8\times 4}$$

$$K_c=(k_1k_2\dots k_8) \qquad \in R^{4\times 8}$$

and introduce the special symbols

$$\hat{k}_7=k_7-\alpha(0\ 0\ 0\ 1)^T$$

$$k_7'=k_7-\alpha(0\ 0\ 1\ 0)^T$$

$$\hat{k}_8=k_8-\beta(0\ 0\ 0\ 1)^T$$

$$k_8'=\bar{k}_8$$

$$L=(u_1+Vk_1 \quad u_2+Vk_2\dots u_6+Vk_6)$$

Then

$$U+VK_c=(L \quad u_7+Vk_7 \quad u_8+Vk_8) \tag{19}$$

If the parameters are on the same row, then using elementary operations for expanding a determinant results in

$$\det(U+VK_c)=\alpha\det(L \quad v_4 \quad u_8+V\hat{k}_8) + \beta\det(L \quad u_7+V\hat{k}_7 \quad v_4)+\det(L \quad u_7+V\hat{k}_7 \quad u_8+V\hat{k}_8) \tag{20}$$

Evaluation of these determinants, for a given value of s, results in two linear equations to be solved for α and β.

If the parameters are on distinct rows, then an expansion of Eq. (18) results in

$$\det(U+VK_c)=\alpha\beta\det(L \quad v_3 \quad v_4)+\alpha\det(L \quad v_3 \quad u_8+Vk_8') + \beta\det(L \quad u_7+Vk_7' \quad v_4)+\det(L \quad u_7+Vk_7' \quad u_8+Vk_8') \tag{21}$$

An evaluation of these determinants, for a given value of s, results in two nonlinear equations to be solved for α and β. In general, there will be two solutions for each value of s.

The success or failure of the numerical technique depends on the analyst's ability to evaluate the determinants of the complex matrices U and V. Since the matrix U is constructed by adding s to the main diagonal of $-A_c+G_cC_c$, a Gaussian elimination properly modified to handle complex elements can be efficiently utilized. The matrix V is also complex and requires the inversion of the X matrix. In general, the order of X will be much larger than the order of V. However, the specific form of the equations allows for an explicit expression for the inverse; this was given by Eq. (7). Thus the computational task is restricted to the size of the estimator plant, and the size of the dynamical plant does not introduce any theoretical difficulties.

A computer code was written in FORTRAN/3000 for an HP-3000 Series II Computer System which solves Eq. (20) or Eq. (21) for the parameters (α,β). As the L matrix in those equations is repeated many times, a modified Gaussian elimination scheme was constructed to work on the ratios of the coefficients. This modification allows one to essentially take only one pass through the Gaussian scheme and reduces the computations considerably. The code is interactive and allows the user to select any two parameters in K_c; the user may elect to exclude the matrix B_c if desired.

The results of a computer run of the parameter-plane technique applied to the design example are portrayed in Fig. 5. The parameters selected for the analysis are elements (1,1) and (1,2) in K_c, i.e., $\alpha=k_{11}$ and $\beta=k_{12}$. Solution of Eq. (20) (in this case) yields the dynamic stability boundaries shown by the continuous contours. The adjacent arrows indicate the direction (along the contours) of increasing ω. Several representative values of ω on the contours are indicated in adjacent parentheses. The portions of the dynamic stability contours that enclose the stable region are indicated by double cross-hatching. Also shown (see Remarks subsection, below) is the induced location of the static stability boundary. It is seen as a dashed contour, with single cross-hatching, parallel to and below the k_{11} axis. The operating point selected by the positivity technique is at (12.3, 162.9). Robustness of this design point is indicated by the size of the stability region enveloping it. For example, if k_{12} is held fixed, then the closed-loop system will be stable if $-105<k_{11}<220$. It might also be observed that the number in parentheses also indicate values of quasistable roots. Thus, for example, if $k_{11}=-105$ and $k_{12}=163$, then $s=j46.7$ is a root of the characteristic equation.

Verification of the technique and numerical calculations was accomplished by recasting the original second-order system, Eqs. (1) and (2), as a system of 40 first-order differential equations. The QR algorithm was then used to obtain eigenvalues for various values of the parameters. These very satisfactory results are also shown in Fig. 5 by squares (with S for stable) or X's (with U for unstable).

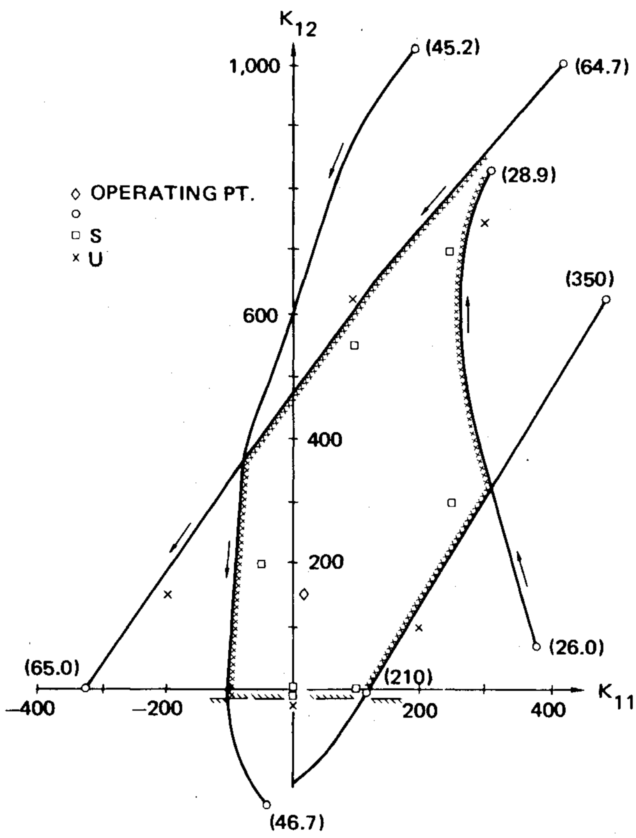

Fig. 5 Parameter plane.

Remarks

Numerical and theoretical difficulties should be expected when the inverse of X does not exist. This will happen if, and only if,

$$s=j\omega_i$$

or

$$u_R(s)b^T(s^2I+\Omega^2)^{-1}b=1$$

The static boundary of a parameter plane occurs when $s=0$ and is usually found by examining the lowest-order coefficient of the characteristic equation. For the example of the paper this term is identically zero so a simple substitution of $s=0$ in the original equations will not help. The authors are currently trying to discover this boundary by manipulating the matrix equations into alternate forms. This boundary does exist as can be seen by the unstable root at $k_{11}=0$ and $k_{12}=-50$ of Fig. 5.

V. Conclusions

A positivity concept has been applied to a moderately high-order system consisting of a model of a flexible satellite plant, a controller, and an estimator. Concurrently, the extended parameter plane concept was applied to determine the region of stability in terms of two selected parameters. It was shown that the design point resulting from the positivity approach lay well within the stable region defined by the parameter-plane approach. In actual design practice, the procedure would be repeated for other selected important pairs of parameters to more fully investigate robustness qualities.

Acknowledgments

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